Solutions to Problem Set 5

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Data Compression With And Without Deep Probabilistic Models
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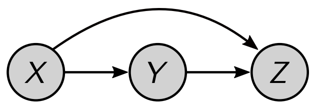
Course materials available at https://robamler.github.io/teaching/compress23/

Problem 5.1: Conditional Independence

In last week's lecture, we learned that every probability distribution P satisfies the so-called chain rule of probability theory. For example, for any three random variables X, Y, and Z, we can always factorize their joint probability distribution as follows (see illustration on the right),



$$P(X, Y, Z) = P(X)\, P(Y \,|\, X)\, P(Z \,|\, X, Y). \tag{1}$$

We then introduced the concept of *conditional (statistical) independence* between two random variables X and Z given a third random variable Y, which is defined analogously to the ordinary (i.e., unconditional) statistical independence as follows,

$$X \text{ and } Z \text{ are conditionally independent given } Y \;:\Leftrightarrow\; P(X, Z \,|\, Y) = P(X \,|\, Y)\, P(Z \,|\, Y). \tag{2}$$

(a) Show that conditional independence between X and Z given Y means that, once you know the value of Y, learning about the value of X would not provide any additional information about Z, i.e.,

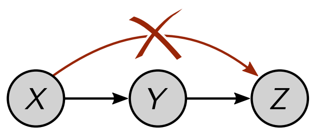
$$X \text{ and } Z \text{ are cond. independ. given } Y \quad \Leftrightarrow \quad P(Z \,|\, X, Y) = P(Z \,|\, Y). \tag{3}$$

Solution: Eq. 3 follows by solving Eq. 2 for $P(Z \,|\, Y)$:

$$\begin{aligned} & P(X, Z \,|\, Y) = P(X \,|\, Y)\, P(Z \,|\, Y) \\ \Leftrightarrow\; & P(Z \,|\, Y) = \frac{P(X, Z \,|\, Y)}{P(X \,|\, Y)} = \frac{P(X, Y, Z)}{P(Y)} \frac{P(Y)}{P(X, Y)} = \frac{P(X, Y, Z)}{P(X, Y)} = P(Z \,|\, X, Y). \end{aligned}$$

$\blacksquare$

Remark: Eq. 3 implies that, if and only if X and Z are conditionally independent given Y, then the chain rule from Eq. 1 simplifies as follows (see illustration on the right),



$$X \text{ and } Z \text{ are cond. indep. given } Y \Leftrightarrow P(X, Y, Z) = P(X)\, P(Y|X)\, P(Z|Y). \tag{4}$$

We refer to the property expressed by Eq. 4 also by saying that X, Y, and Z form a *Markov chain* $X \to Y \to Z$. A Markov chain can be interpreted as a memoryless stochastic process: if you want to draw a random sample from a Markov chain, then you can proceed as follows: first, draw a random sample $x \sim P(X)$, then draw $y \sim P(Y \,|\, X{=}x)$, and finally draw $z \sim P(Z \,|\, Y{=}y)$. Notice that, once you've drawn y, you no longer need to keep x in memory because you won't need it for drawing z.

Markov chains play an important role in information theory since communication pipelines can typically be modeled as chains of memoryless stages, where each stage transforms the communicated data into some new representation. We'll meet Markov chains again when we discuss channel coding and lossy compression, and you'll prove an important bound on how information propagates along a Markov chain—the so-called data processing inequality—on Problem Set 10.

Comparison to ordinary independence: we now show that conditional independence is neither a stronger nor a weaker property than ordinary statistical independence.

(b) Show that two random variables X and Z can be statistically independent even if they are *not* conditionally independent given some third random variable Y.

Hint: Consider our Simplified Game of Monopoly. You already showed in Problem 4.1 (b) that X_{red} and X_{blue} are statistically independent. Now show that X_{red} and X_{blue} are, however, *not* conditionally independent given X_{sum}.

Solution: By definition, conditional independence holds if and only if the two probability distributions on the left and right-hand sides of Eq. 2 are equal. Two probability distributions are equal if they assign the same probabilities to all possible inputs. Thus, in order to show that X_{red} and X_{blue} are *not* conditionally independent given X_{sum}, we only have to find a single triple of values x_{red}, x_{blue}, and x_{sum} for which

$$\begin{aligned} &P(X_{\text{red}}{=}x_{\text{red}}, X_{\text{blue}}{=}x_{\text{blue}} \,|\, X_{\text{sum}}{=}x_{\text{sum}}) \\ &\qquad \neq P(X_{\text{red}}{=}x_{\text{red}} \,|\, X_{\text{sum}}{=}x_{\text{sum}})\, P(X_{\text{blue}}{=}x_{\text{blue}} \,|\, X_{\text{sum}}{=}x_{\text{sum}}). \end{aligned}$$

You can easily find many examples for x_{red}, x_{blue}, and x_{sum} for which this is the case. For example, we have

$$P(X_{\text{red}}{=}1, X_{\text{blue}}{=}1 \,|\, X_{\text{sum}}{=}3) = 0$$

but, according to our example in the lecture notes for Lecture 4,

$$P(X_{\text{red}}{=}1 \,|\, X_{\text{sum}}{=}3)\, P(X_{\text{blue}}{=}1 \,|\, X_{\text{sum}}{=}3) = \frac{1}{2} \times \frac{1}{2} \neq 0.$$

Intuitively, this makes sense: the red and the blue die are thrown independently of each other, but if we're told their sum then the equation $X_{\text{red}} + X_{\text{blue}} = X_{\text{sum}}$ introduces a constraint that ties X_{red} and X_{blue} together. ∎

(c) Show that two random variables X and Z can be conditionally independent given some third random variable Y even if X and Z are *not* statistically independent.

Hint: Any (nontrivial) Markov process $X \to Y \to Z$ will do: conditioning on Y "cuts" the dependency between X and Z. For example, consider a sequence of three coin tosses and let X, Y, and Z be the number of times that the coin comes up "heads" in the first, the first two, and all three tosses, respectively. Find an expression for $P(Z \,|\, X, Y)$ without being overly formal (think about the experimental setup and the interpretation of conditional probability rather than its formal mathematical definition). Then convince yourself that X and Z are conditionally independent given Y by Eq. 3. Show by providing a counter example that, without conditioning on Y, then X and Z are *not* statistically independent.

Solution: Assuming a fair coin for simplicity, we have

$$P(Z \,|\, X, Y) = \begin{cases} \frac{1}{2} & \text{if } Z \in \{Y, Y+1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Here, the fact that the right-hand side does not depend on X means that conditioning on X is unnecessary, i.e., $P(Z \,|\, X, Y) = P(Z \,|\, Y)$ and thus X and Z are conditionally independent given Y by Eq. 3. However, without conditioning on Y, we have, e.g.,

$$P(X\!=\!1, Z\!=\!0) = 0 \qquad \text{but} \qquad P(X\!=\!1)P(Z\!=\!0) = \frac{1}{2} \times \frac{1}{8} \neq 0.$$

Thus, X and Z are *not* statistically independent. ■

Problem 5.2: Expressiveness of Probabilistic Models

In the lecture, we introduced various model architectures to efficiently approximate complicated probability distributions. Let us now analyze how expressive each of these architectures is. In particular, we analyze whether each of the proposed architecture can model *correlations* between symbols in a message, i.e., the fact that, in messages that appear in the real world, symbols are typically *not* statistically independent. All models below describe a message $\mathbf{X} = (X_1, X_2, \ldots, X_k)$ where each symbol X_i, $i \in \{1, 2, \ldots, k\}$ is modeled as a random variable with values from some discrete alphabet $\mathfrak{X}$.

The four parts (a)-(d) of this problem can be solved independently. So don't give up if you have troubles solving one of the parts.

(a) **Fully factorized models:** before we look at more complicated model architectures below, let's consider the most trivial model architecture, which assumes that all symbols X_i, $i \in \{1, 2, \ldots, k\}$ are statistically independent. Such a model is often called "fully factorized" because the joint probability distribution $P(\mathbf{X})$ of the message $\mathbf{X}$ can be written as a product of the marginal distributions:

$$P_{\text{model}}(\mathbf{X}) = \prod_{i=1}^{k} P_{\text{model}}(X_i). \tag{5}$$

Here, we reinstated the subscript "model" because we want to search for the best model, $P^*_{\text{model}}(\mathbf{X})$, that can be written in the form of Eq. 5 and that best approximates some data distribution $P_{\text{data}}(\mathbf{X})$, which is typically *not* fully factorized.

(i) Consider the cross entropy $H\big(P_{\text{data}}(\mathbf{X}), P_{\text{model}}(\mathbf{X})\big)$. Convince yourself that, for a model of the form of Eq. 5 (warning: but not for more general models),

$$H\big(P_{\text{data}}(\mathbf{X}), P_{\text{model}}(\mathbf{X})\big) = \sum_{i=1}^{k} H\big(P_{\text{data}}(X_i), P_{\text{model}}(X_i)\big) \quad \text{(if Eq. 5 holds)} \tag{6}$$

where $P_{\text{data}}(X_i)$ is the marginal distribution of X_i under P_{data} (i.e., the distribution that you obtain if you *marginalize* $P_{\text{data}}(\mathbf{X})$ over all X_j with $j \neq i$).

Solution: We simply write out the cross entropy on the left-hand side of Eq. 6, use linearity of the expectation, and then marginalize each term over all X_j with $j \neq i$. For your reference, the following calculation is very elaborate; you weren't expected to write it out in such detail:

$$\begin{aligned}
H\big(P_{\text{data}}(\mathbf{X}), P_{\text{model}}(\mathbf{X})\big) &= \mathbb{E}_{P_{\text{data}}(\mathbf{X})}\big[-\log_2 P_{\text{model}}(\mathbf{X})\big] \\
&= \mathbb{E}_{P_{\text{data}}(\mathbf{X})}\left[-\sum_{i=1}^{k} \log_2 P_{\text{model}}(X_i)\right] \\
&= -\sum_{i=1}^{k} \mathbb{E}_{P_{\text{data}}(\mathbf{X})}\big[\log_2 P_{\text{model}}(X_i)\big] \\
&\overset{(*)}{=} -\sum_{i=1}^{k} \left(\sum_{(X_1,\ldots,X_k)\in\mathfrak{X}^k} P_{\text{data}}(X_1,\ldots,X_k) \times \log_2 P_{\text{model}}(X_i) \right) \\
&\overset{(\triangle)}{=} -\sum_{i=1}^{k} \left(\sum_{X_i\in\mathfrak{X}} P_{\text{data}}(X_i) \times \log_2 P_{\text{model}}(X_i) \right) \\
&= -\sum_{i=1}^{k} \mathbb{E}_{P_{\text{data}}(X_i)}\big[\log_2 P_{\text{model}}(X_i)\big] \\
&= \sum_{i=1}^{k} H\big(P_{\text{data}}(X_i), P_{\text{model}}(X_i)\big)
\end{aligned}$$

Where, in the equality marked with "$(*)$", we explicitly write out the expectation over $\mathbf{X} = (X_1, \ldots, X_k)$, and in the equality marked with "$(\triangle)$", we marginalize over all X_j with $j \neq i$. ■

(ii) Argue that the right-hand side of Eq. 6 is minimized by setting $P^*_{\text{model}}(X_i) = P_{\text{data}}(X_i)$ for all i. Thus, within the class of fully factorized models (Eq. 5), the best approximation $P^*_{\text{model}}(\mathbf{X})$ of an arbitrary distribution $P_{\text{data}}(\mathbf{X})$ is the product of the marginals, $P^*_{\text{model}}(\mathbf{X}) = \prod_{i=1}^{k} P_{\text{data}}(X_i)$.

Hint: what is the cross entropy $H(P, P)$ of a distribution with itself, and why is it smaller or equal than any $H(P, Q)$ for all other distributions $Q \neq P$?

Solution: We first note that the cross entropy of a distribution with itself is just the normal entropy, $H(P, P) = H[P]$. Thus, choosing any other $P'_{\text{model}}(X_i) \neq P_{\text{data}}(X_i)$ would increase the cross entropy by

$$H\big(P_{\text{data}}(X_i), P'_{\text{model}}(X_i)\big) - H\big[P_{\text{data}}(X_i)\big] = D_{\text{KL}}\big(P_{\text{data}}(X_i) \,||\, P'_{\text{nodel}}(X_i)\big) \geq 0.$$

■

(iii) Convince yourself that, for this optimal fully factorized model, the cross entropy (and thus the expected bit rate) is the sum of the marginal entropies of all symbols under the data distribution,

$$H\big(P_{\text{data}}(\mathbf{X}), P^*_{\text{model}}(\mathbf{X})\big) = \sum_{i=1}^{k} H_{P_{\text{data}}}(X_i) \qquad \text{(if Eq. 5 holds).} \tag{7}$$

Solution: Inserting $P^*_{\text{model}}(X_i) = P_{\text{data}}(X_i)$ into the right-hand side of Eq. 6 and using $H(P, P) = H[P]$ leads to Eq. 7. ■

(b) **Markov Chains:** as discussed in the lecture, a Markov chain models the creation of a sequence of symbols $X_1, X_2, \ldots, X_k$ as a memoryless stochastic process, i.e.,

$$P(\mathbf{X}) = P(X_1) \prod_{i=2}^{k} P(X_i \,|\, X_{i-1}) \tag{8}$$

where, from here on, we drop the subscript "model" for simplicity.

(i) Show that, although each symbol X_i is conditioned only on its immediately preceding symbol X_{i-1} (for $i > 1$) and not on any earlier symbols, a Markov chain can still model correlations between *any* symbols, not just nearest neighbors. More specifically, show that there exists a model of the form of Eq. 8 where two symbols X_i and X_j are *not* statistically independent for at least *some* $i, j \in \{1, \ldots, k\}$ with $j \geq i + 2$.

Hint: For example, you could consider the Markov chain over the alphabet $\mathfrak{X} = \{0, 1\}$ with $P(X_1{=}0) = P(X_1{=}1) = \frac{1}{2}$ and

$$P(X_i \,|\, X_{i-1}) = \begin{cases} 0.99 & \text{if } X_i = X_{i-1}; \\ 0.01 & \text{if } X_i \neq X_{i-1}. \end{cases} \tag{9}$$

Describe in words what a random sample $\mathbf{x} \sim P(\mathbf{X})$ from this model would typically look like. Then convince yourself (either by explicit calculation or by less formal and more intuitive arguments) that all marginal probabilities are $P(X_i\!=\!0) = P(X_i\!=\!1) = \frac{1}{2}\ \forall i$ by symmetry but that, e.g., the conditional probability $P(X_j\!=\!1\,|\,X_i\!=\!1) > \frac{1}{2}$ for at least *some* non-neighboring i, j (it turns out to be true for *all* i, j, but this is more difficult to show formally).

Solution: The probabilities in Eq. 9 were deliberately chosen this dramatic so as to point you to an interpretation of the model: the model describes sequences of bits, where one typically has long runs of identical bits before the bit flips. Therefore, while we have $P(X_i\!=\!0) = P(X_i\!=\!1) = \frac{1}{2}$ for each individual bit X_i by symmetry, any two bits X_i and X_j are more likely to be equal than unequal, especially if $|i-j|$ is not too large. This is easy to show formally for symbols $j > i$ that are not too far away from each other:

$$P(X_j\!=\!1\,|\,X_i\!=\!1) \geq P(X_{i+1}\!=\!1, X_{i+2}\!=\!1, \ldots, X_j\!=\!1\,|\,X_i\!=\!1) = 0.99^{j-i}$$

which is larger than $P(X_j = 1) = \frac{1}{2}$ as long as $j - i \leq 68$. Therefore, X_i and X_j are not statistically independent in all these cases.

Note: The equation above only states a *lower bound* on $P(X_j\!=\!1\,|\,X_i\!=\!1)$, but that's enough to prove that there exist some non-neighboring pairs of symbols that are not statistically independent. From our interpretation of Eq. 9, we'd expect that *no* pairs of symbols are statistically independent in this model; they only become *close* to being independent with growing distance $\delta := j - i$ (i.e., $\lim_{\delta\to\infty} I_P(X_i; X_{i+\delta}) = 0$). This is in fact true: using the so-called transfer matrix method, which is out of scope for this problem set, one finds that $P(X_j\!=\!1\,|\,X_i\!=\!1) = \frac{1}{2}\big(1 + 0.98^{|i-j|}\big) > \frac{1}{2}\ \forall i, j$.) ■

(ii) Now show that, although a Markov chain can model symbols that are not statistically independent, any two symbols X_i and X_l with $l \geq i + 2$ are *conditionally* independent given any X_j with $i < j < l$.

Hint: write out the joint probability of all symbols *up to* X_l as follows,

$$P(\mathbf{X}) = \underbrace{\left(P(X_1)\prod_{\alpha=2}^{i} P(X_\alpha|X_{\alpha-1})\right)}_{=P(X_1,\ldots,X_i)} \underbrace{\left(\prod_{\alpha=i+1}^{j} P(X_\alpha|X_{\alpha-1})\right)}_{=P(X_{i+1},\ldots,X_j|X_i)} \underbrace{\left(\prod_{\alpha=j+1}^{l} P(X_\alpha|X_{\alpha-1})\right)}_{=P(X_{j+1},\ldots,X_l|X_j)}. \tag{10}$$

What do you get if you now marginalize both sides over all symbols except X_i, X_j, and X_l? Compare the result to Eq. 4.

Solution: Marginalizing both sides of Eq. 10 over all symbols except X_i, X_j, and X_l results in

$$P(X_i, X_j, X_l) = P(X_i)\,P(X_j\,|\,X_i)\,P(X_l\,|\,X_j)$$

which is precisely of the form of Eq. 4. ■

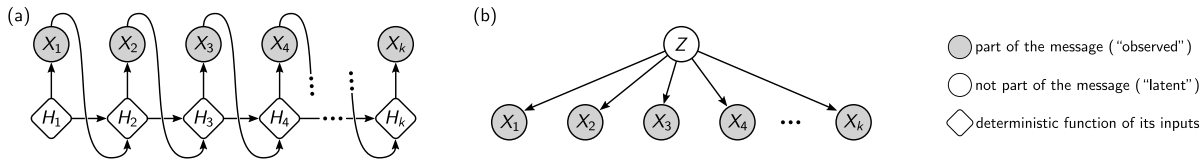


Figure 1: (a) autoregressive model, see Problem 5.2 (c); (b) latent variable model, see Problem 5.2 (d)

(c) **Autoregressive models:** Figure 1 (a) illustrates an autoregressive model like the one you've used in Problem 3.2. The figure is a graphical representation of the following factorization of the joint probability distribution,

$$P(\mathbf{X}) = \prod_{i=1}^{k} P(X_i \,|\, H_i) \qquad \text{with} \qquad H_1 = \text{fixed};\; H_{i+1} = f(H_i, X_i) \tag{11}$$

where f is some deterministic function (e.g., a neural network). Show that autoregressive models are more powerful than Markov chains in that they can model probability distributions where two symbols X_i and X_l are *not* conditionally independent given some third symbol X_j with $i < j < l$.

Hint: For example, you could consider a toy autoregressive model over the alphabet $\mathfrak{X} = \{0, 1\}$ with $H_1 = 0$ and $H_{i+1} = f(H_i, X_i) = (H_i + X_i) \bmod 10$. Thus, the hidden state H_i counts how many "1" symbols have appeared before symbol X_i (modulo 10 so that the hidden states don't grow out of bounds). Now you could make the probability of "1" symbols depend on H_i, e.g., by setting $P(X_i{=}1 \,|\, H_i) = \frac{H_i+1}{10}$ and $P(X_i{=}0 \,|\, H_i) = 1 - \frac{H_i+1}{10}$. Then, consider the first three symbols X_1, X_2, and X_3 (the statement is also true for other triples of symbols, but the calculations are more tedious). Show by explicit calculation that

$$P(X_3{=}1 \,|\, X_1{=}1, X_2{=}1) \neq P(X_3{=}1 \,|\, X_2{=}1), \tag{12}$$

i.e., that even this simple toy model already violates the right-hand side of Eq. 3. The value of the left-hand side of Eq. 12 follows directly from unrolling the model but calculating the right-hand side takes a few more steps. Before you do these calculations, test your understanding by reasoning in words whether you expect the left-hand side of Eq. 12 to be smaller or larger than the right-hand side.

Solution: Since every "1"-bit increases the probability of subsequent "1"-bits, we expect the left-hand side of Eq. 12 to be larger than the right-hand side. Let's check this by explicit calculation.

To evaluate the left-hand side of Eq. 12, we can simply unroll the autoregressive model until the point where it models the symbol X_3. Since H_i counts how many "1" symbols have appeared before symbol X_i (modulo 10), we get $H_3 = 2$ and therefore $P(X_3{=}1 \,|\, X_1{=}1, X_2{=}1) = \frac{3}{10} = 0.3$.

To evaluate the right-hand side of Eq. 12, we explicitly write out the conditional probability and then express both enumerator and denominator as a marginalization over X_1,

$$P(X_3\!=\!1 \,|\, X_2\!=\!1) = \frac{P(X_2\!=\!1, X_3\!=\!1)}{P(X_2\!=\!1)} = \frac{\sum_{x_1 \in \mathfrak{X}} P(X_1\!=\!x_1, X_2\!=\!1, X_3\!=\!1)}{\sum_{x_1 \in \mathfrak{X}} P(X_1\!=\!x_1, X_2\!=\!1)}$$
$$= \frac{\frac{9}{10}\frac{1}{10}\frac{2}{10} + \frac{1}{10}\frac{2}{10}\frac{3}{10}}{\frac{9}{10}\frac{1}{10} + \frac{1}{10}\frac{2}{10}} = \frac{24}{110} \approx 0.218$$

which is indeed smaller than the left-hand side, as we expected. ■

(d) **Latent variable models:** Figure 1 (b) illustrates a latent variable model. You'll learn how to use latent variable models for effective data compression with the so-called bits-back trick in Lecture 7. But let's first prove here that latent variable models can in fact capture correlations between symbols.

The illustration in Figure 1 (b) is a pictorial representation of the following factorization of a joint probability distribution over symbols $\mathbf{X} = (X_1, \ldots, X_k)$ and a (usually multidimensional) so-called *latent* variable Z,

$$P(\mathbf{X}, Z) = P(Z) \prod_{i=1}^{k} P(X_i \,|\, Z). \tag{13}$$

Here $P(Z)$ is called the "prior distribution" and $P(X_i \,|\, Z)$ is called the "likelihood". At a first glance, the model architecture in Eq. 13 might look like it couldn't possibly capture any correlations between different symbols X_i because the part of Eq. 13 that describes symbols is fully factorized (similar to the model in Eq. 5). However, this impression is deceptive because the symbols X_i are only *conditionally independent given the latent* Z. However, Z is not part of the message. The probabilistic model of the message is the *marginal* distribution of $\mathbf{X}$,

$$P(\mathbf{X}) = \begin{cases} \sum_Z P(\mathbf{X}, Z) & \text{for discrete } Z; \\ \int P(\mathbf{X}, Z)\, dZ & \text{for continuous } Z. \end{cases} \tag{14}$$

Show that the marginal distribution in Eq. 14 can indeed describe correlations between symbols, i.e., a distribution of this form can model data sources where any two symbols X_i and X_l are *not* statistically independent, and are also *not* conditionally independent given any different third symbol X_j.

Hint: You could consider, e.g., a toy model over the alphabet $\mathfrak{X} = \{0, 1\}$ with $k = 3$, boolean $Z \in \{0, 1\}$, and with a likelihood $P(X_i \,|\, Z)$ that is the same for all i. Come up with some explicit probabilities for $P(Z\!=\!z)$ and $P(X_i\!=\!x_i \,|\, Z\!=\!z)$ for all $z, x_i \in \{0, 1\}$. Then show first that $P(X_1\!=\!x_1, X_3\!=\!x_3) \neq P(X_1\!=\!x_1)\, P(X_3\!=\!x_3)$ and finally that $P(X_3\!=\!x_3 \,|\, X_1\!=\!x_1, X_2\!=\!x_2) \neq P(X_3\!=\!x_3 \,|\, X_2\!=\!x_2)$ in your model for some $x_1, x_2, x_3 \in \{0, 1\}$ of your choice. Try to explain your findings in words too: why does knowing the value of, e.g., X_1 influence the probability distribution over X_3?

Solution: Let's use a uniform prior $P(Z)$ for simplicity and a likelihood $P(X_i|Z)$ that favors X_i to be equal to the latent variable Z. Thus, let $\alpha > \frac{1}{2}$ and

$$P(Z\!=\!z) = \frac{1}{2} \;\; \forall z \in \{0,1\} \qquad \text{and} \qquad P(X_i\!=\!x_i \,|\, Z\!=\!z) = \begin{cases} \alpha & \text{if } x_i = z; \\ 1-\alpha & \text{if } x_i \neq z. \end{cases}$$

It is generally a good idea to reason informally about extreme cases before doing formal calculations. Here, the extreme case is where α is almost one. Since both the prior probability and the likelihood remain unchanged if we simultaneously flip the latent bit Z and all symbols X_i, each individual symbol X_i is either "0" or "1" with equal probability, $P(X_i\!=\!0) = P(X_i\!=\!1) = \frac{1}{2}$. However, for $\alpha \approx 1$, we expect that most symbols X_i are equal to Z and thus, *even if we don't know* Z, we can predict that most symbols are probably equal to each other. Or, put in different words, if we know, e.g., that $X_1 = 1$, then the most probable explanation for this is that $Z = 1$, which would then also make it probable that $X_3 = 1$. Conversely, if we know that $X_1 = 0$, then the most probable explanation for this is that $Z = 0$, which would then also make it probable that $X_3 = 0$. Thus, $P(X_3 \,|\, X_1)$ depends on X_1 and therefore X_1 and X_3 are not statistically independent. If we now also know the value of X_2 then we have even more evidence to reason about Z and, consequently, the probability of X_3 changes again.

More formally, we have, e.g.,

$$\begin{aligned} P(X_i\!=\!1) &= \sum_{z\in\{0,1\}} P(Z\!=\!z, X_i\!=\!1) = \sum_{z\in\{0,1\}} P(Z\!=\!z)\, P(X_i\!=\!1 \,|\, Z\!=\!z) \\ &= \frac{1}{2} \times \alpha + \frac{1}{2} \times (1-\alpha) = \frac{1}{2} \end{aligned}$$

and therefore

$$P(X_1\!=\!1)\, P(X_3\!=\!1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

whereas, for $i \neq j$ we have, according to our model in Eqs. 13-14,

$$\begin{aligned} P(X_i\!=\!1, X_j\!=\!1) &= \sum_{z\in\{0,1\}} P(Z\!=\!z, X_i\!=\!1, X_j\!=\!1) \\ &= \sum_{z\in\{0,1\}} P(Z\!=\!z)\, P(X_i\!=\!1 \,|\, Z\!=\!z)\, P(X_j\!=\!1 \,|\, Z\!=\!z) \\ &= \frac{1}{2}\big[(1-\alpha)^2 + \alpha^2\big] = \frac{1}{2} - \alpha + \alpha^2 = \frac{1}{4} + \left(\alpha - \frac{1}{2}\right)^2 \\ &> \frac{1}{4} \qquad \forall \alpha \neq \frac{1}{2} \end{aligned}$$

which proves that X_i and X_j are not statistically independent.

To prove that X_1 and X_3 are also not conditionally independent given X_2 we show that they don't form a Markov chain, i.e., we use Eq. 3 and show that $P(X_3\!=\!1 \,|\, X_1\!=\!1, X_2\!=\!1) \neq P(X_3\!=\!1 \,|\, X_2\!=\!1)$. For simplicity, we chose a concrete value of $\alpha = 0.9$ here. We obtain the right-hand side by combining the above results,

$$P(X_3\!=\!1 \,|\, X_2\!=\!1) = \frac{P(X_2\!=\!1, X_3\!=\!1)}{P(X_2\!=\!1)} = \frac{\frac{1}{2}\big[(1-\alpha)^2 + \alpha^2\big]}{\frac{1}{2}} = 0.82$$

whereas, for the left-hand side,

$$\begin{aligned} P(X_3\!=\!1 \,|\, X_1\!=\!1, X_2\!=\!1) &= \frac{P(X_1\!=\!1, X_2\!=\!1, X_3\!=\!1)}{P(X_1\!=\!1, X_2\!=\!1)} \overset{(*)}{=} \frac{\frac{1}{2}\big[(1-\alpha)^3 + \alpha^3\big]}{\frac{1}{2}\big[(1-\alpha)^2 + \alpha^2\big]} \\ &= \frac{0.73}{0.82} \approx 0.89 > P(X_3\!=\!1 \,|\, X_2\!=\!1) \end{aligned}$$

where the equality marked with "$(*)$" expresses both the enumerator and the denominator again as a marginalization over $Z \in \{0, 1\}$. ■