Problem Set 12

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Data Compression With And Without Deep Probabilistic Models
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Course materials available at https://robamler.github.io/teaching/compress23/

Note: This week's tutorial will be replaced by Lucas Theis' talk. The two short problems below are meant to be discussed in small groups in the lecture before the talk.

Problem 12.1: Recovering the Lossless Limit

In the last lecture, we considered a lossless compression pipeline $\mathbf{X} \to \mathbf{S} \to \mathbf{X}'$, and we stated that the expected bit rate is bounded by the rate/distortion curve,

$$\mathbb{E}_P[\text{bit rate}] \geq \mathcal{R}(\mathcal{D}) \qquad \text{with} \qquad \mathcal{R}(\mathcal{D}) := \inf_{\substack{P(\mathbf{X}'|\mathbf{X}): \\ \mathbb{E}_P[d(\mathbf{X},\mathbf{X}')] \leq \mathcal{D}}} I_P(\mathbf{X};\mathbf{X}'). \tag{1}$$

Here, $d(\mathbf{x}, \mathbf{x}') \geq 0$ quantifies how much a reconstruction $\mathbf{x}'$ differs from the original message $\mathbf{x}$, and $\mathcal{D}$ specifies how much distortion we accept in expectation.

What do you get for $\mathcal{R}(\mathcal{D})$ in the limit of lossless compression, $\mathcal{D} = 0$, assuming that $d(\mathbf{x}, \mathbf{x}') = 0$ if and only if $\mathbf{x} = \mathbf{x}'$? Interpret your result.

Problem 12.2: The Noisy Parking Disk

This problem is meant to provide some intuition for the optimal channel coders we constructed in the last lecture. The problem is an adaptation of the "noisy typewriter" example from the MacKay book (see link on the course website).

We defined the capacity C of a memoryless channel $P(\mathbf{Y} \,|\, \mathbf{X}) = \prod_{i=1}^{k} P(Y_i \,|\, X_i)$,

$$C := \sup_{P(X_i)} I_P(X_i; Y_i). \tag{2}$$

Consider a memoryless channel where both the inputs $X_i \in \mathcal{X}$ and the outputs $Y_i \in \mathcal{Y}$ are integers from one to twelve, i.e., $\mathcal{X} = \mathcal{Y} = \{1, 2, \ldots, 12\}$. Picture these twelve numbers arranged in a circle, like they are on an analog clock or a parking disc. Transmitting a symbol $x_i \in \mathcal{X}$ goes as follows: the sender points to the number x_i on the circle, and the receiver reads off the indicated number as y_i. Unfortunately, the sender has very thick fingers, and therefore the receiver might confuse the indicated number with one of its immediate neighbors. More precisely,

$$P(Y_i = y_i \,|\, X_i = x_i) = \begin{cases} \frac{1}{3} & \text{if } y_i \in \{x_i \ominus 1, x_i, x_i \oplus 1\} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where "$\ominus$" and "$\oplus$" denote subtraction and addition that wraps around the circle.

(a) Show that the channel capacity is $C = 2$ bits.

Hint: express the mutual information as $I_P(X_i; Y_i) = H_P(Y_i) - H_P(Y_i|X_i)$. Why does it suffice to maximize only $H_P(Y_i)$? What is the maximum entropy $H_P(Y_i)$ of a random variable $Y_i \in \mathcal{Y}$? Notice that you don't need to find the optimal input distribution $P(X_i)$ to derive the capacity C here.

(b) Show that one possible input distribution that maximizes $I_P(X_i; Y_i)$ in Eq. 2 is a uniform distribution, i.e., $P(X_i = x_i) = \frac{1}{12}\ \forall x_i \in \mathcal{X}$

(c) While a uniform input distribution $P(X_i = x_i) = \frac{1}{12}\ \forall x_i \in \mathcal{X}$ does maximize the mutual information $I_P(X_i; Y_i)$, designing a channel code that uses all possible input values $x_i \in \mathcal{X}$ is somewhat difficult in practice. Luckily, the uniform distribution is not the only input distribution that maximizes the mutual information for the noisy parking disc channel. Can you come up with some very simple channel encoder $P(\mathbf{X} \mid \mathbf{S})$ and channel decoder $P(\mathbf{S}' \mid \mathbf{Y})$ that admit perfect reconstruction of all possible inputs $\mathbf{s} \in \{0, 1\}^k$, and that allow you to transmit exactly 2 bits per channel invocation?

Hint: You don't need any fancy theorems here. Just think simple: how can you avoid ambiguities on the receiver side given the specific form of the channel in Eq. 3?